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LETTER TO THE EDITOR

Algebraic invariants of the O(2) gauge transformation

J H H Perk†, F Y Wu‡ and X N Wu‡

† Department of Physics, Oklahoma State University, Stillwater, OK 74078, USA
‡ Department of Physics, Northeastern University, Boston, MA 02115, USA

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Abstract. We consider the O(2) gauge transformation for a two-state vertex model on a lattice, and derive its fundamental algebraic invariants, the minimal set of homogeneous polynomials of the vertex weights which are invariant under O(2) transformations. Explicit expressions of the fundamental invariants are given for symmetric vertex models on lattices with coordination number $p = 2, 3, 4, 5, 6$, generalising $p = 3$ results obtained previously from more elaborate considerations.

In a study of the symmetry properties of discrete spin systems, Wegner [1] introduced a gauge transformation generalising the weak-graph transformation used by earlier investigators [2–4]. The gauge transformation, which describes important symmetry properties including the usual duality relation [3], is a linear transformation of the weights of a vertex model under which the partition function remains invariant. One particular symmetry property studied for over a century [5] is the construction of algebraic invariants, the homogeneous polynomials invariant under linear transformations. The problem of constructing invariants for the gauge transformation in vertex models has been studied by Hijmans *et al* [6,7] for the square lattice and, more recently, by Wu *et al* [8] and by Gwa [9] for the O(2) transformation on trivalent lattices. Specifically, Wu *et al* [8] proposed that the critical frontier of the Ising model in a non-zero magnetic field is given by the algebraic invariants of the related vertex model, and constructed the invariants by enumeration for trivalent lattices. A simpler method leading to the same invariants was later given by Gwa [9]. But the extension of both of these analyses to lattices of general coordination number p has proven to be extremely tedious, becoming almost intractable for $p > 4$. Clearly, an alternative and simpler approach is needed.

In this letter we consider the O(2) gauge transformation for a two-state vertex model on a lattice of general coordination number p, and present a formulation which leads to a simple and direct determination of its algebraic invariants.

We first define the vertex model and the O(2) gauge transformation. Consider a lattice of coordination number p, with the lattice edges in one of two distinct states independently at each edge. We may regard the edges as being either 'empty' or 'covered' by a bond, so that the edge configurations generate bond graphs [10]. Introduce edge variables $s = 0, 1$ so that $s = 0$ ($s = 1$) denotes the edge being empty (covered). With each lattice site associate a vertex weight $W(s_1, s_2, \ldots, s_p)$, where $s_1, s_2, \ldots, s_p$ indicate the states of the p incident edges. The partition function of this two-state vertex model is

$$Z(\{W\}) = \sum \prod_i W(s_1, s_2, \ldots, s_p) \tag{1}$$

where the summation is taken over all bond graphs of the lattice, and the product is taken over all vertices i.

Consider a linear transformation of the 2^p vertex weights $W(s_1, s_2, \ldots, s_p)$,

$$\tilde{W}(t_1, t_2, \ldots, t_p) = \sum_{s_1=0}^{1} \sum_{s_2=0}^{1} \ldots \sum_{s_p=0}^{1} R_{t_1s_1} R_{t_2s_2} \ldots R_{t_ps_p} W(s_1, s_2, \ldots, s_p). \tag{2}$$

The transformation (2) leaves the partition function invariant if R_{ts} are elements of a 2×2 matrix $\mathbf{R}$ satisfying $\tilde{\mathbf{R}}\mathbf{R} = I$, where I is the identity matrix [1]. This implies $\det|R_{ts}| = \pm1$, and therefore the transformation (2) provides a representation of the two-dimensional orthogonal group O(2), to be referred to as the O(2) gauge transformation.

The O(2) group is generated by a rotation $\mathbf{R}^{(1)}$ or a reflection $\mathbf{R}^{(2)}$ given by

$$\mathbf{R}^{(1)} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \qquad \mathbf{R}^{(2)} = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}. \tag{3}$$

Note that $\mathbf{R}^{(2)}$ has been used exclusively in previous investigations [2,4,8].

For *symmetric* vertex models, the vertex weights depend only on the number of covered incident edges, for which we have

$$W(s_1, s_2, \ldots, s_p) = W_p(s) \qquad s = s_1 + s_2 + \ldots + s_p \tag{4}$$

where $s = 0, 1, 2 \ldots, p$ is the number of bonds incident at the vertex. We shall, however, continue to assume general vertex weights, and only below specialise the results to symmetric vertex models.

Hilbert [5, see also p 235 of Gurevich in [5]] established more than a century ago that invariants of a linear transformation are in the form of homogeneous polynomials, and that all such polynomials are expressible in terms of a minimal set of *fundamental* ones. The crux of the matter is, of course, the determination of these fundamental invariants for a given linear transformation. For the O(2) transformation, as we now show, the task can be accomplished as follows.

Introduce the change of basis

$$A_{\sigma_1\ldots\sigma_p} = \sum_{t_1=0}^{1} \ldots \sum_{t_p=0}^{1} \left[\prod_{k=1}^{p} (\mathrm{i}\sigma_k)^{t_k}\right] W(t_1, \ldots, t_p) \tag{5}$$

where $\sigma_k = \pm1$. For example, for $p = 2$, (5) is

$$\begin{aligned} A_{\pm\pm} &= [W(00) - W(11)] \pm \mathrm{i}[W(01)] + W(10)] \\ A_{\pm\mp} &= [W(00) + W(11)] \mp \mathrm{i}[W(01) - W(10)]. \end{aligned} \tag{6}$$

In a similar fashion we define $\tilde{A}_{\sigma_1}\ldots\sigma_p$ in terms of $\tilde{W}(s_1, \ldots, s_p)$. Then, using the identity

$$\sum_{t=0}^{1} (\mathrm{i}\sigma)^t R_{ts}^{(l)} = (-1)^{s(l-1)} \mathrm{e}^{\mathrm{i}\sigma\theta} (\mathrm{i}\sigma)^s \qquad l = 1, 2 \tag{7}$$

where $R_{ts}^{(l)}$ are elements of $\mathbf{R}^{(1)}$ or $\mathbf{R}^{(2)}$ given by (3), one obtains from (5) and (2) the following transformation property for the A:

$$\tilde{A}^{(l)}_{\sigma_1\ldots\sigma_p} = \begin{cases} \mathrm{e}^{i(\sigma_1+\ldots+\sigma_p)\theta} A_{\sigma_1\ldots\sigma_p} & l = 1 \\ \mathrm{e}^{i(\sigma_1+\ldots+\sigma_p)}\theta A^*_{\sigma_1\ldots\sigma_p} & l = 2 \end{cases} \tag{8}$$

where $\tilde{A}^{(l)}$ is the $\tilde{A}$ obtained from (5) by using $\tilde{W}$ in (2) with $\mathbf{R}=\mathbf{R}^{(l)}$, $l=1,2$, and A^* is the complex conjugate of A. For $l=1$, the As change only by a phase factor under the O(2) transformation. It follows that any product of the As, for which the σs of all A factors in the product add to zero, is an invariant. For $l=2$, however, the products are transformed into complex conjugates, in addition to the change of a phase factor. Thus, in both cases the real parts of these products are invariant, and the imaginary parts are invariant under $\mathbf{R}^{(1)}$ while changing sign under $\mathbf{R}^{(2)}$. We shall, however, refer to both the real and imaginary parts as the invariants.

For a given p, the fundamental invariants can now be constructed by following the above prescription. For $p=2$, there are three fundamental invariants $A_{++}A_{--}$, A_{+-}, and A_{-+}, the last two being the complex conjugates of each other; any other invariant is composed of the three. For $p=3$, our consideration leads to 30 distinct products, examples of which are $A_{+++}A_{---}$, $A_{++-}A_{+--}$, $A^2_{++-}A_{+-+}A_{---}$, and $A^3_{++-}A_{---}$. This gives rise to 30 homogeneous polynomials which are invariant under $\mathbf{R}^{(1)}$, and either invariant or changing sign under $\mathbf{R}^{(2)}$.

It is shown above that fundamental invariants for the general O(2) gauge transformation (2) can be constructed in a straightforward fashion. We now specialise the consideration to the symmetric vertex model (4) for which the situation is considerably simpler.

Using (4) and (5), we have

$$A_{\sigma_1\ldots\sigma_p} \equiv A_p(t) = \sum_{s=0}^{p} \mathrm{i}^s W_p(s) \sum_{\alpha=0}^{m} \binom{m}{\alpha} \sum_{\beta=0}^{n} \binom{n}{\beta} (-1)^\beta \delta_{\mathrm{Kr}}(\alpha+\beta, s) \tag{9}$$

where $t\equiv\sigma_1+\sigma_2\ldots+\sigma_p=\pm p, \pm(p-2),\ldots,\pm1$ or 0; $m\equiv(p+t)/2$, $n\equiv(p-t)/2$, and δ_{Kr} is the Kronecker delta function. Note that the coefficient of $W_p(s)$ in (9) is the coefficient of z^s in the expansion of $(1+\mathrm{i}z)^m(1-\mathrm{i}z)^n$. It follows that (8) becomes

$$\tilde{A}^{(l)}_{\sigma_1\ldots\sigma_p} \equiv \tilde{A}^{(l)}_p(t) = \begin{cases} \mathrm{e}^{\mathrm{i}t\theta} A_p(t) & l=1 \\ \mathrm{e}^{\mathrm{i}t\theta} A^*_p(t) & l=2. \end{cases} \tag{10}$$

It is straightforward to write down the explicit expressions of $A_p(t)$ using (9). Adopting the notation [4,8] of denoting vertex weights by $a, b, c, \ldots$ such that a is the weight of vertices having no incident bonds, b the weight of vertices having one incident bond, etc, we find for $p=2,3,\ldots,6$

$$\begin{aligned}
&A_2(\pm2)=a-c\pm2\mathrm{i}b && A_2(0)=a+c\\
&A_3(\pm3)=a-3c\pm\mathrm{i}(3b-d) && A_3(\pm1)=a+c\pm\mathrm{i}(b+d)\\
&A_4(\pm4)=a-6c+e\pm4\mathrm{i}(b-d) && A_4(\pm2)=a-e\pm2\mathrm{i}(b+d)\\
&A_4(0)=a+2c+e\\
&A_5(\pm5)=a-10c+5e\pm\mathrm{i}(5b-10d+f)\\
&A_5(\pm3)=a-2c-3e\pm\mathrm{i}(3b+2d-f)\\
&A_5(\pm1)=a+2c+e\pm\mathrm{i}(b+2d+f)\\
&A_6(\pm6)=a-15c+15e-g\pm2\mathrm{i}(3b-10d+3f)\\
&A_6(\pm4)=a-5c-5e+g\pm4\mathrm{i}(b-f)\\
&A_6(\pm2)=a+c-e-g\pm2\mathrm{i}(b+2d+f)\\
&A_6(0)=a+3c+3e+g.
\end{aligned} \tag{11}$$

As dictated by (10), the fundamental invariants for each p are now constructed by forming products of $A_p(t)$, such that the sum of all t variables in the product vanishes. Adopting the notation $(t)\equiv A_p(t)$ for each p, we find the following fundamental invariants for $p=2,3,\ldots,6$:

$p=2$: $(2)(-2)$; (0)

$p=3$: $(3)(-3)$; $(1)(-1)$; $(3)(-1)^3, (-3)(1)^3$

$p=4$: $(4)(-4)$; $(2)(-2)$; (0); $(4)(-2)^2, (-4)(2)^2$

$p=5$: $(5)(-5)$; $(3)(-3)$; $(1)(-1)$; $(5)(-1)^5, (-5)(1)^5$; $(3)(-1)^3, (-3)(1)^3$; $(5)(-3)(-1)^2, (-5)(3)(1)^2$; $(5)(1)(-3)^2, (-5)(-1)(3)^2$; $(5)^2(-1)(-3)^3, (-5)^2(1)(3)^3$; $(5)^3(-3)^5, (-5)^3(3)^5$

$p=6$: $(6)(-6)$; $(4)(-4)$; $(2)(-2)$; (0); $(6)(-2)(-4), (-6)(2)(4)$; $(4)(-2)^2, (-4)(2)^2$; $(6)(-2)^3, (-6)(2)^3$; $(6)(2)(-4)^2, (-6)(-2)(4)^2$; $(6)^2(-4)^3, (-6)^2(4)^3$.

Since (t) and $(-t)$ are complex conjugates of each other, the fundamental invariants always occur in conjugate pairs (except (0) and $(t)(-t)$ which are real), and we can consider the real and imaginary parts individually. Both the real and imaginary parts are invariant under $\mathbf{R}^{(1)}$, and the real parts are invariants and the imaginary parts change sign under $\mathbf{R}^{(2)}$. For $p=3$, e.g., there are four polynomials:

$$
\begin{aligned}
I_1 &= (3)(-3) = (a-3c)^2+(3b-d)^2 \\
I_2 &= (1)(-1) = (a+c)^2+(b+d)^2 \\
I_3 &= \mathrm{Re}(3)(-1)^3 = \mathrm{Re}\{[a-3c+\mathrm{i}(3b-d)][a+c-\mathrm{i}(b+d)]^3\} \\
I_4 &= \mathrm{Im}(3)(-1)^3 = \mathrm{Im}\{[a-3c+\mathrm{i}(3b-d)][a+c-\mathrm{i}(b+d)]^3\}
\end{aligned} \tag{12}
$$

for which I_1, I_2, and I_3 are invariant under both $\mathbf{R}^{(1)}$ and $\mathbf{R}^{(2)}$, and I_4 is invariant under $\mathbf{R}^{(1)}$ while changing sign under $\mathbf{R}^{(2)}$.

We have considered the O(2) gauge transformation for a general two-state vertex model on an arbitrary lattice, and constructed its fundamental algebraic invariants. For the symmetric vertex model on a lattice of coordination number p, our analysis shows that there are, respectively, 2, 4, 5, 15, and 14 fundamental algebraic invariants for $p=2$, 3, 4, 5, and 6. These invariants are explicitly given in (11). In the case of $p=3$, we have verified that the fundamental invariants P, Q, P_1, P_2 and P_3 obtained previously [8,9] can indeed be expressed in terms of those in (12). The relations are $P=(9I_2-I_1)/8$, $Q=(I_1-I_2)/8$, $P_1=I_4/4$, $P_2=(72I_3-I_1^2+30I_1I_2+27I_2^2)/64$, and $P_3=(-8I_3-I_1^2+6I_1I_2+3I_2^2)/64$. Note that P_1 changes sign under $\mathbf{R}^{(2)}$, a fact previously observed [8]. For $p=4$, we have also verified that the five fundamental invariants deduced from the ones obtained by Hijmans *et al* [6,7] can be expressed in terms of those in (11). The same set of $p=4$ fundamental invariants have also been obtained, after considerable algebraic manipulation, by extending the analyses of Wu *et al* [8] and Gwa [9], but the method of Gwa no longer retains its simplicity for $p=3$. Both methods, however, become almost intractable for $p>4$.

Finally, we point out the existence of *syzygies*, polynomial relations between the linearly independent invariants. We have seen that all invariants for a given p are products of $p+1$ polynomials $A_p(t)$. It follows that there must exist relations, or syzygies, among these invariants, if the number of invariants exceeds p. Explicit expressions of syzygies are usually very difficult to construct, but they are easily identified in the present formulation. For $p=3$ and 4, e.g., the numbers of fundamental invariants are, respectively, 4 and 5, and hence there is one syzygy in each case. Explicitly, we find

$$\begin{aligned} &[(3)(-1)^3][(-3)(1)^3]=[(3)(-3)][(1)(-1)]^3 && \text{for } p=3 \\ &[(4)(-2)^2][(-4)(2)^2]=[(4)(-4)][(2)(-2)]^2 && \text{for } p=4. \end{aligned} \tag{13}$$

Similarly, there are ten syzygies for $p=5$ and eight for $p=6$; all can be similarly constructed.

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